Advanced Higher Christmas Problems

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Questions

1. Rudolph's Nose

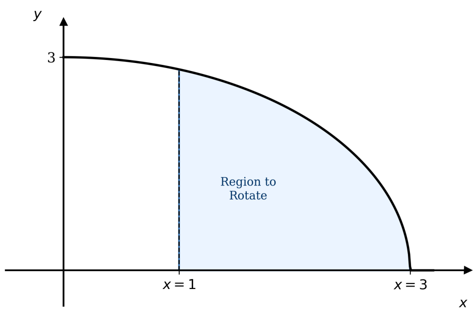

Figure 1: Diagram of the region bounded by $y = \sqrt{9 - x^2}$ and $x = 1, x = 3$.

Rudolph's nose is modelled by rotating the curve $y = \sqrt{9 - x^2}$ about the x-axis, between the limits $x = 1$ and $x = 3$. Calculate the exact volume of the solid generated.

2. Elf and Safety

The Toy Factory has two automated supply lines that drop parts into a single packing chute.

- **Line A** drops a Train Body every 825 seconds.
- **Line B** drops a Wheel Set every 315 seconds.

The chute requires exactly 10 seconds to clear a part.

- If parts arrive **simultaneously** (Gap = 0), an emergency breaker trips.
- If parts arrive **separately but within 10 seconds** of each other, the machine jams.

1. Use the Euclidean Algorithm to find the **smallest non-zero time gap** (GCD) that can occur between drops. Use this to determine if the machine is at risk of jamming.

2. The Safety Elf wants to observe this "closest call" to calibrate the sensors. Find integers x and y such that $825x + 315y = \text{GCD}$.

3. **Interpretation:** Explain what the values of x and y represent in terms of the machine's cycles.

3. The Reindeer Trumpet

Father Christmas has a special reindeer-calling trumpet with 5 valves. To play a specific chord, he must press down a specific set of valves simultaneously.
Calculate the number of ways he can press down exactly 3 valves at once.

4. Sleigh Stability

The stability of the sleigh is given by the roots of the polynomial:

$$z^3 - 3z^2 + 4z - 12 = 0$$

1. Find all roots of the equation ($z \in \mathbb{C}$).

2. If a system is unstable when any root has a **positive real part**, is the sleigh stable?

5. Christmas Decoration

A decoration for the top of the tree is designed based on the complex roots of $z^5 = 32$.

1. Solve the equation $z^5 = 32$ for z. Express your answers in polar form: $r(\cos\theta + i\sin\theta)$.

2. Draw the decoration by representing your answer and an Argand diagram.

6. Elvish Accounting

The elves at the North Pole use Base 12 (digits: $0, 1, \ldots, 9, A, B$) as their number system.

1. Convert the Base 10 number **2025** into Base 12.

2. Convert the Base 12 number **B3** into Base 10.

7. Sleigh Velocity

The sleigh accelerates according to $a(t) = 6t - \cos(t)$. Given it starts from rest ($v = 0$) at the origin ($s = 0$) when $t = 0$:

1. Find an expression for velocity $v(t)$.

2. Calculate the displacement of the sleigh at time $t = \pi$.

Solutions

1. Volume:

$$V = \pi \int_1^3 y^2 \, dx = \pi \int_1^3 (9 - x^2) \, dx$$

$$V = \pi \left[9x - \frac{x^3}{3} \right]_1^3 = \pi \left((27 - 9) - (9 - \frac{1}{3}) \right)$$

$$V = \pi \left(18 - 8\frac{2}{3} \right) = \pi \left(\frac{54}{3} - \frac{26}{3} \right) = \frac{28\pi}{3} \text{ units}^3$$

The solid would look like this:

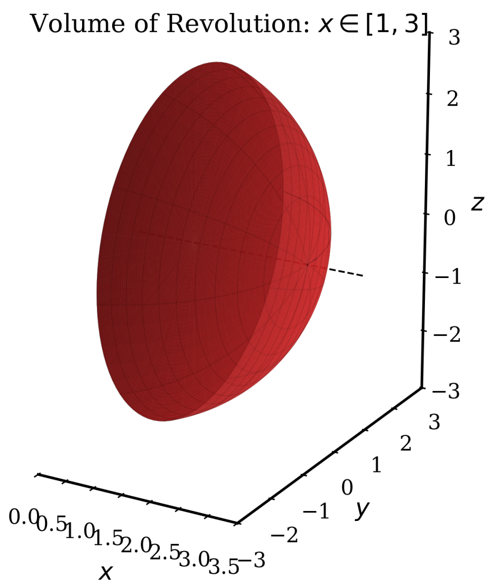

Figure 2: The solid formed by rotating $y = \sqrt{9 - x^2}$ about the x-axis.

2. Elf and Safety

1. Finding the Minimum Gap (GCD): Using the Euclidean Algorithm on 825 and 315:

$$\begin{aligned}
825 &= 2(315) + 195 \\
315 &= 1(195) + 120 \\
195 &= 1(120) + 75 \\
120 &= 1(75) + 45 \\
75 &= 1(45) + 30 \\
45 &= 1(30) + 15 \\
30 &= 2(15) + 0
\end{aligned}$$

The GCD is **15 seconds**. **Conclusion:** Apart from simultaneous arrivals (where the breaker trips), the closest the parts will ever get is 15 seconds. Since the chute clears in 10 seconds ($15 > 10$), the machine is **SAFE** from jamming.

2. Linear Combination: Working backwards:

$$
\begin{aligned}
15 &= 45 - 30 \\
15 &= 45 - (75 - 45) = 2(45) - 75 \\
15 &= 2(120 - 75) - 75 = 2(120) - 3(75) \\
15 &= 2(120) - 3(195 - 120) = 5(120) - 3(195) \\
15 &= 5(315 - 195) - 3(195) = 5(315) - 8(195) \\
15 &= 5(315) - 8(825 - 2(315)) \\
15 &= 21(315) - 8(825)
\end{aligned}
$$

Rearranging to $825x + 315y$:

$$825(-8) + 315(21) = 15$$

So, $\mathbf{x} = -\mathbf{8}$ and $\mathbf{y} = \mathbf{21}$.

3. Interpretation: The "closest call" happens when Line B completes its **21st cycle** and Line A completes its **8th cycle**. At this moment, the parts arrive exactly 15 seconds apart.

3. The Reindeer Trumpet:

Calculate the number of ways he can press down exactly 3 valves at once.

We need to choose 3 valves out of 5, where the order does not matter. This is calculated using ${}^5C_3$:

$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(2 \times 1)}$$

$$\binom{5}{3} = \frac{120}{6 \times 2} = \frac{120}{12} = 10$$

There are 10 ways to press exactly 3 valves.

4. Complex Roots:

By inspection $z = 3$ is a root: $27-27+12-12 = 0$. Divide by $(z-3)$: $z^2(z-3)+4(z-3) = (z^2+4)(z-3) = 0$. Roots: $z = 3, z = 2i, z = -2i$. Stability: The root $z = 3$ has a positive real part. **Unstable.**

5. Christmas Decoration

Solve the equation $z^5 = 32$ for z.

First, express 32 in polar form. The modulus is 32 and the argument is 0.

$$z^5 = 32(\cos 0 + i \sin 0)$$

Using de Moivre's Theorem, the roots are given by $z_k = \sqrt[5]{32}\left(\cos \frac{0+2k\pi}{5} + i \sin \frac{0+2k\pi}{5}\right)$ for $k = 0, 1, 2, 3, 4$. Since $\sqrt[5]{32} = 2$, the roots are:

$$\begin{aligned}
k=0: &\quad z_0 = 2(\cos 0 + i\sin 0) \\
k=1: &\quad z_1 = 2\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right) \\
k=2: &\quad z_2 = 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right) \\
k=3: &\quad z_3 = 2\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right) \\
k=4: &\quad z_4 = 2\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right)
\end{aligned}$$

2. Argand diagram:

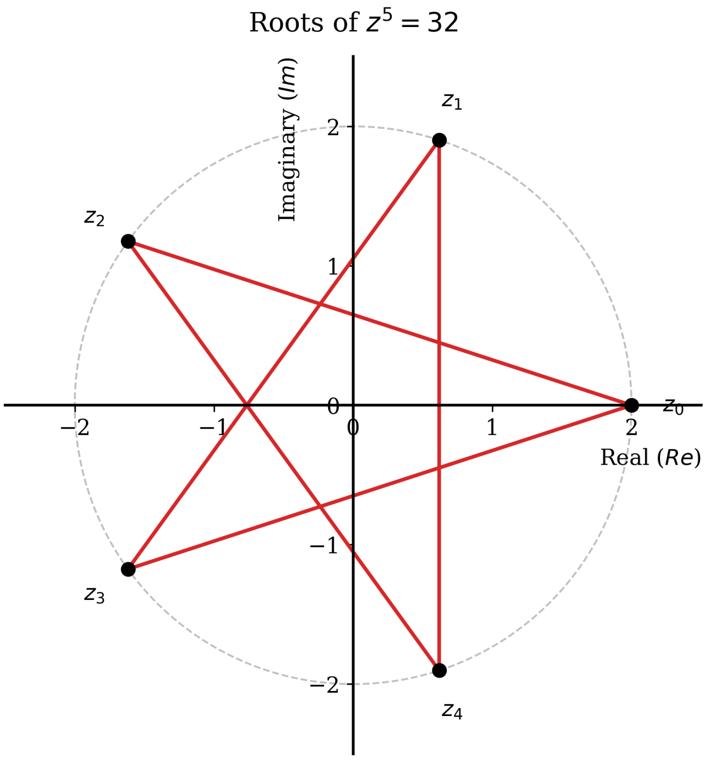

Figure 3: The roots of $z^5 = 32$ form a regular pentagon.

6. Elvish Accounting

1. **Convert** 2025_{10} **to Base 12:**

$$\begin{aligned}
2025 \div 12 = 168 &\quad \text{remainder } \mathbf{9} \\
168 \div 12 = 14 &\quad \text{remainder } \mathbf{0} \\
14 \div 12 = 1 &\quad \text{remainder } \mathbf{2} \\
1 \div 12 = 0 &\quad \text{remainder } \mathbf{1}
\end{aligned}$$

Reading the remainders from bottom to top:

$$\text{Result} = \mathbf{1209_{12}}$$

2. **Convert $B3_{12}$ to Base 10:**

   Recall that $B = 11$.

   $$B3_{12} = 11(12^1) + 3(12^0) = 132 + 3 = \mathbf{135}$$

7. Sleigh Velocity

1. **Find an expression for velocity $v(t)$.**

   We integrate the acceleration $a(t)$ with respect to t:

   $$v(t) = \int (6t - \cos t)\,dt = 3t^2 - \sin t + C_1$$

   Using the initial condition $v(0) = 0$:

   $$0 = 3(0)^2 - \sin(0) + C_1 \implies C_1 = 0$$

   Thus, the expression for velocity is:

   $$v(t) = 3t^2 - \sin t$$

2. **Calculate the displacement of the sleigh at time $t = \pi$.**

   We integrate the velocity $v(t)$ to find displacement $s(t)$:

   $$s(t) = \int (3t^2 - \sin t)\,dt = t^3 + \cos t + C_2$$

   Using the initial condition $s(0) = 0$:

   $$0 = (0)^3 + \cos(0) + C_2 \implies 0 = 1 + C_2 \implies C_2 = -1$$

   The expression for displacement is $s(t) = t^3 + \cos t - 1$.

   Substituting $t = \pi$:

   $$s(\pi) = \pi^3 + \cos(\pi) - 1$$
   $$s(\pi) = \pi^3 - 1 - 1$$
   $$s(\pi) = \pi^3 - 2$$

Questions by Bryn Jones, Newbattle High School. Find more resources at Applying Maths.

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Permalink: https://applyingmaths.com/assets/pdfs/advanced-higher-christmas-2025.pdf